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AN ANALYSIS OF STEADY FULLY DEVELOPED HEAT TRANSFER IN LAMINAR FLOW WITH VISCOUS DISSIPATION IN A CURVED CIRCULAR DUCT

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Abstract—In this paper a problem of steady state fully developed heat transfer in laminar flow of constant physical properties in a curved tube, effected by a heat source (or sink) distribution present in wall material, has been considered. Thermal boundary conditions have been discussed for a curved tube of circular cross-section, taking into consideration the heat generation due to viscous dissipation. A perturbation analysis has been carried out for those situations in which the normal temperature gradient at the inner wall surface is regarded as prescribed and constant circumferentially. Solutions for average Nusselt number and the rate of change of fluid temperature in the main flow direction have been obtained and discussed. Effects of viscous dissipation phenomenon on these quantities have been investigated. In the discussions, curved and straight duct cases have been compared.

NOMENCLATURE

a,	cross-sectional radius, Fig. 1;
b,	radius of curvature, Fig. 1;
c_p,	specific heat at constant pressure;
K,	coefficient of thermal conductivity;
Nu,	Nusselt number;
P,	Prandtl number;
$\bar{r}, \alpha, \theta$,	co-ordinate system, Fig. 1;
R,	Reynolds number, based on longitudinal pressure gradient;
T,	local temperature;
U, V, W,	velocity components in directions of increasing $\bar{r}, \alpha, \theta$ respectively;
ρ, μ,	density and coefficient of viscosity respectively;
λ,	a/b.

Subscripts

a,	mean value along cross-sectional boundary;
m,	mean value over cross-sectional domain;
0,	straight duct case with same cross-sectional radius, longitudinal pressure gradient and wall normal temperature gradient as in curved duct;
w,	value at solid–fluid interface.

1. INTRODUCTION

1.1

DEAN [1, 2] initiated theoretical studies of flows of fluids in curved tubes. Heat-transfer problems associated with such flows constitute a modern area of research. The following problem is of both engineering and academic interest.

1.2 *Physical problem*

A Newtonian fluid flows through a curved tube of circular cross-section. The flow is laminar. The wall material (but not the fluid) contains a uniform heat source (or sink) distribution. Both velocity and temperature fields are fully developed and steady. The following conditions (which are usual assumptions) are satisfied.

(i) Area of cross-section and radius of curvature remain constant in longitudinal direction.
(ii) Variations of physical properties are negligible.
(iii) Secondary free convection effects are negligible.

1.3

When the conditions (ii) and (iii) are satisfied, viscous dissipation may be non-negligible. This point has been discussed, on the basis of the dynamical similarity principle, in [3, 4]. In these and in [5], all of which have dealt with the above physical problem for straight

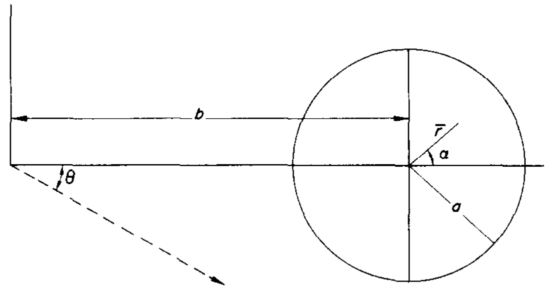

FIG. 1. Coordinate system.

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ducts, qualitatively interesting and quantitatively significant effects of viscous dissipation have been reported. The effects of this physical phenomenon in a curved duct should necessarily be different from those in a straight duct, since the velocity distribution in one is quantitatively and qualitatively different from that in the other. Therefore, it is of interest to know how these effects are altered when one proceeds from the straight to the curved shape.

It may be said that the heat generation due to viscous dissipation is considerable in fast laminar flow through narrow tube (such tube flows are found in compact heat exchangers), and also considerable in highly viscous flows (e.g. flows of oils).

1.4

For the thermally fully developed flow region, it is simple to conclude (both for curved and straight ducts) that in the longitudinal direction the normal temperature gradient at the solid–fluid interface remains constant and the temperature at this interface and within the flowing medium varies linearly at a fixed rate. The following discussion draws the remaining picture of thermal boundary conditions.

In a curved tube of circular cross-section, the velocity distribution and the viscous dissipation are circumferentially non-uniform. These, therefore, allow the normal temperature gradient to vary around the cross-sectional boundary, and allow a non-uniform circumferential temperature gradient to exist in the wall material. The latter, in turn, can give rise to the process of conduction in the solid material. Consequently, involvement of the wall thermal conductivity can occur in prescribing the thermal boundary condition circumferentially. In this event, although the cross-sectional boundary is circular and the wall heat source (sink) distribution is uniform, the problem becomes quite complicated. The situations in which the said involvement is either absent or insignificant are described below.

Let the coefficient of thermal conductivity of the wall be infinite (i.e. very large). This makes the circumferential temperature differences in wall material (but not the variation of normal temperature gradient along cross-sectional boundary) to disappear. Thus the prescribed thermal boundary condition is (A): solid–fluid interface temperature is constant and varies linearly in circumferential and longitudinal directions respectively. A similar situation, where (A) is prescribed, is that of wall of infinite (i.e. of very large) thickness. It may be said that if both thermal conductivity coefficient and thickness of wall are large, the boundary condition is prescribed as (A) approximately.

On the other hand, let the coefficient of thermal conductivity of wall be zero (i.e. very small). In this situation, heat flow is not allowed to take place in the wall material. Obviously, wall heat source (sink) distribution can exist only in the inner surface. In case of heat source, the heat generated in each element of inner surface of wall will totally flow in the normal direction to the adjacent fluid. The reverse of this will occur in the case of heat sink. Therefore, it is the normal temperature gradient, not the temperature, which is prescribed along cross-sectional boundary. Thus when wall heat source (sink) distribution is uniform, the solid–fluid interface temperature continues to be circumferentially non-uniform, and the prescribed thermal boundary condition is (B): normal temperature gradient at solid–fluid interface is constant in both circumferential and longitudinal directions. A similar situation, where (B) is prescribed, is that of wall of very small thickness. It may be stated that if both thermal conductivity coefficient and thickness of wall are small, the prescribed boundary condition is nearly as in (B).

In the case of straight duct of circular cross-section, both velocity distribution and viscous dissipation are circumferentially constant. Therefore, when wall heat source (sink) distribution is constant, there are only two boundary conditions, namely, (A) and (B) which are satisfied simultaneously (i.e. if (A) is prescribed (B) holds good and vice versa).

To summarize, there is a marked distinction between thermal boundary condition cases of straight and curved tubes when the cross-sections of both are circular. Such a distinction exists between the cases of straight ducts of circular and non-circular cross-sections discussed in [6, 7]. It is of interest to note that the cases of straight duct of non-circular cross-section and curved duct of circular cross-section are analogous as far as thermal boundary conditions are concerned.

1.5

The theoretical papers which have specifically dealt with the problem stated in Section 1.2 are: Mori and Nakayama [8], Ozisik and Topakoglu [9], Akiyama and Cheng [10] and Kalb and Seader [11]. In each of these papers, only the boundary condition (A) has been considered, and the viscous dissipation has not been taken into account. The first two papers, [8, 9], contain analytical studies, viz. boundary layer and second order perturbation analyses for large and small Dean number values respectively. The remaining two [10, 11], are numerical studies, where the Dean number values not considered in [8, 9] have also been dealt with.

Herein, the case of the boundary condition (B) is analyzed by taking viscous dissipation into consideration.

2. ANALYSIS

2.1

The following energy equation may be considered with regard to the temperature distribution in the problem stated in Section 1.2:

$$\rho c_p\left(U\frac{\partial T}{\partial \bar r}+\frac{V}{\bar r}\frac{\partial T}{\partial \alpha}+\frac{W}{b+\bar r\cos\alpha}\frac{\partial T}{\partial \theta}\right)$$
$$=K\left\{\left(\frac{\partial}{\partial \bar r}+\frac{\cos\alpha}{b+\bar r\cos\alpha}\right)\frac{\partial T}{\partial \bar r}+\left(\frac{1}{\bar r}\frac{\partial}{\partial \alpha}-\frac{\sin\alpha}{b+\bar r\cos\alpha}\right)\frac{1}{\bar r}\frac{\partial T}{\partial \alpha}+\frac{1}{\bar r}\frac{\partial T}{\partial \bar r}\right\}$$
$$+\mu\left[2\left\{\left(\frac{\partial U}{\partial \bar r}\right)^2+\left(\frac{1}{\bar r}\frac{\partial V}{\partial \alpha}+\frac{U}{\bar r}\right)^2+\left(\frac{U\cos\alpha}{b+\bar r\cos\alpha}-\frac{V\sin\alpha}{b+\bar r\cos\alpha}\right)^2\right\}\right.$$
$$+\left(\frac{1}{\bar r}\frac{\partial W}{\partial \alpha}+\frac{W\sin\alpha}{b+\bar r\cos\alpha}\right)^2+\left(\frac{\partial W}{\partial \bar r}-\frac{W\cos\alpha}{b+\bar r\cos\alpha}\right)^2$$
$$\left.+\left(\frac{\partial V}{\partial \bar r}-\frac{V}{\bar r}+\frac{1}{\bar r}\frac{\partial U}{\partial \alpha}\right)^2\right] \quad (1)$$

where the terms involving μ are due to viscous dissipation.

The boundary condition (B) may be stated mathematically as

$$\frac{\partial T}{\partial \bar r}=\beta \quad \text{at solid–fluid interface} \quad (2)$$

where β is constant and is positive (negative) if wall contains heat source (sink) distribution.

Dean in his beautiful work [2] introduced a set of rules to simplify equations of motion and continuity. The resulting equations are approximate (see (A.1) to (A.4) in the Appendix). Any result based on those equations suffers from insignificant error when $(a/b)=\lambda \ll 1$; which is supported by experimental studies [12, 13]. Dean [2] solved the approximate equations by a perturbation method and reported a second order solution for velocity distribution. In [2], the qualitative predictions based on the second order perturbation solution are physically consistent. Dean's analysis [2] has been extended to various hydrodynamic problems in curved tubes, viz. non-Newtonian flows [14]; dispersion of a solute in fluid flow [15], etc.

An analysis similar to that in [2] is carried out for (1) and (2).

According to Dean's simplification rules [2], it is required to replace

$$b+\bar r\cos\alpha \quad \text{by} \quad b \quad (3)$$

$$\frac{\partial}{\partial \bar r}\left(\text{or, } \frac{1}{\bar r}\frac{\partial}{\partial \alpha}\right)\pm\frac{\cos\alpha(\text{or, } \sin\alpha)}{b+\bar r\cos\alpha} \quad \text{by} \quad \frac{\partial}{\partial \bar r}\left(\text{or, } \frac{1}{\bar r}\frac{\partial}{\partial \alpha}\right) \quad (4)$$

in (1) and also to take care that if terms like $(\partial W/\partial \bar r)\{W\cos\alpha/(b+\bar r\cos\alpha)\}$ are being neglected then terms of same order like $(V/\bar r)(\partial V/\partial \bar r)$ are also to be neglected. As a result, (1) reduces to the following very simple form:

$$\rho c_p\left(U\frac{\partial T}{\partial \bar r}+\frac{V}{\bar r}\frac{\partial T}{\partial \alpha}+\frac{W}{b}\frac{\partial T}{\partial \theta}\right)=K\left(\frac{\partial^2 T}{\partial \bar r^2}+\frac{1}{\bar r}\frac{\partial T}{\partial \bar r}+\frac{1}{\bar r^2}\frac{\partial^2 T}{\partial \alpha^2}\right)+\mu\left[\left(\frac{\partial W}{\partial \bar r}\right)^2+\left(\frac{1}{\bar r}\frac{\partial W}{\partial \alpha}\right)^2\right]. \quad (5)$$

Equations (5) and (2) are transformed to following dimensionless forms:

$$\nabla^2 t=P\left(\frac{1}{r}\frac{\partial(\phi,t)}{\partial(\alpha,r)}+wm\right)-\bar B\left[\left(\frac{\partial w}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial w}{\partial \alpha}\right)^2\right] \quad (6)$$

$$\frac{\partial t}{\partial r}=1 \quad \text{at solid–fluid interface} \quad (7)$$

where ∇^2 is Laplacian operator in (r,α); first term in the coefficient of P involves Jacobian notation; and

$$r=\frac{\bar r}{a},\ w=\frac{\rho a W}{\mu},\ \frac{1}{r}\frac{\partial \phi}{\partial \alpha}=\frac{\rho a U}{\mu},\ \frac{\partial \phi}{\partial r}=-\frac{\rho a V}{\mu}, \quad (8)$$

$$m=\frac{M}{\beta},\ M=\frac{1}{b}\frac{\partial T}{\partial \theta}=\frac{1}{b}\frac{\partial T_w}{\partial \theta}=\frac{1}{b}\frac{\mathrm{d}\,T_m}{\mathrm{d}\,\theta}, \quad (9)$$

$$t=\frac{T-T_m}{a\beta}, \quad (10)$$

$$\bar B=\frac{\mu^3}{\rho^2 a^3 K\beta}. \quad (11)$$

Last two equations of (8) define secondary flow stream function $\phi(r,\alpha)$. In (9), M is independent of $\bar r,\alpha$ and θ. The dimensionless temperature difference t, which is defined by (10), is independent of θ. It may be noted that the definition (10) is specifically appropriate for the case of boundary condition (B).

While solving (6) under (7), the relations:

$$t_m=0 \quad (12)$$

and

$$t\neq\infty \quad (13)$$

are to be used in order to obtain bounded unique solution for t, where (12) is derived from (10), and (13) means that the temperature in the duct remains finite.

2.2

Out of two coupled differential equations in ϕ and w (see (A.5) and (A.6) in the Appendix), the one corresponding to secondary flow contains λ and the other involves Reynolds number R. Therefore λ is to be used as perturbation parameter.

Introducing the series:

$$\phi=\lambda\phi_1+\lambda^2\phi_2+\ldots,\ w=w_0+\lambda w_1+\lambda^2 w_2+\ldots \quad (14)$$

in the equations governing ϕ and w, deriving the governing differential equations for the coefficient functions in (14) and solving those equations under the conditions of no-slip flow at solid–fluid interface and finite velocity within duct, it is found that

$$w_0 = w_0(r), w_1 = w_{11}(r)\cos\alpha,$$
$$w_2 = w_{21}(r)+w_{22}(r)\cos 2\alpha \quad (15)$$
$$\phi_1 = \phi_{11}(r)\sin\alpha, \phi_2 = \phi_{22}(r)\sin 2\alpha \quad (16)$$

where

$$w_0 = R(1-r^2), \quad (17)$$

$$w_{11} = \frac{R^3}{11\,520}(19r-40r^3+30r^5-10r^7+r^9), \quad (18)$$

$$w_{21} = \frac{R^5}{112(5760)^2}(-4119+21\,280r^2-46\,340r^4 +55\,440r^6-39\,830r^8+17\,584r^{10} -4620r^{12}+640r^{14}-35r^{16}) \quad (19)$$

and

$$\phi_{11} = \frac{R^2}{288}(4r-9r^3+6r^5-r^7). \quad (20)$$

2.3

In conformity with (14), t is expanded as

$$t = t_0+\lambda t_1+\lambda^2 t_2+\ldots. \quad (21)$$

This and (7) yield

$$\frac{dt_0}{dr} = 1, \frac{\partial t_1}{\partial r} = 0, \frac{\partial t_2}{\partial r} = 0, \ldots \text{at solid–fluid interface.} \quad (22)$$

In the case of boundary condition (B), m (i.e. M) is supposed to be unknown. Further, m is expected to assume different values in curved and straight ducts. Hence, when t is given by (21), m is to be expressed as

$$m = m_0+\lambda m_1+\lambda^2 m_2+\ldots. \quad (23)$$

Mathematically, two conditions should accompany equation (6) because (6) is of second order. There are three such conditions, viz. (7), (12) and (13). This means one of these three conditions is meant for evaluation of some quantity. That quantity can be m only. In fact, the primary use of (7) is to determine m. As will be shown in the sequel, the only use of first and third equations of (22) is to determine m_0 and m_2. Hence, the above supposition, i.e. the series expansion in (23), is mathematically consistent.

Inserting (14), (21) and (23) in (6) and equating coefficients of $\lambda^i (i = 0, 1, 2, \ldots)$, there results a system of differential equations. First three equations of that system are successively cited and solved.

The first equation is

$$\nabla^2 t_0 = P m_0 w_0 - \bar{B}(dw_0/dr)^2 \quad (24)$$

where t_0 does not depend upon α. Integrating both sides of (24) over cross-sectional domain, applying Gauss Theorem on l.h.s. and using first equation of (22), m_0 is found as

$$m_0 = \frac{4}{RP}(1+B) \quad (25)$$

where B, which is to be called as dissipation number, is given by

$$B = \bar{B}R^2. \quad (26)$$

When (24) is solved after inserting in it the result (25), two constants of integration occur which are determined by the conditions in (13) and (12). The solution is found as

$$t_0 = \frac{1}{12}(-5+12r^2-3r^4) + \frac{B}{6}(-2+6r^2-3r^4). \quad (27)$$

The second equation is the following partial differential equation

$$\nabla^2 t_1 = P\left(\frac{1}{r}\frac{\partial\phi_1}{\partial\alpha}\frac{dt_0}{dr} + m_0 w_1 + m_1 w_0\right) -2\bar{B}\frac{dw_0}{dr}\frac{\partial w_1}{\partial r}. \quad (28)$$

When both sides of (28) are integrated over cross-sectional domain and Gauss Theorem and second equation of (22) are used on l.h.s., m_1 is found as

$$m_1 = 0. \quad (29)$$

When expressions of m_1 and others are inserted in (28), the resulting equation admits t_1 in the form of

$$t_1 = t_{11}(r)\cos\alpha \quad (30)$$

and then reduces to second order ordinary differential equation with t_{11} as unknown. Solving that equation and determining the integration constants by using $(dt_{11}/dr)_{r=1} = 0$, which is derived from (30) and second equation of (22), and using (13), the following solution is found:

$$t_{11} = \frac{R^2}{34\,560}\left\{\frac{1}{10} \times (-256r+285r^3-200r^5+75r^7-15r^9+r^{11}) + \frac{P}{2}(-161r+240r^3-220r^5+105r^7-24r^9+2r^{11}) + B(11r+57r^3-80r^5+45r^7-12r^9+r^{11}) + BP(-68r+120r^3-130r^5 +75r^7-21r^9+2r^{11})\right\}. \quad (31)$$

The third equation is the following partial differential equation

$$\nabla^2 t_2 = P\left\{\frac{1}{r}\frac{\partial(\phi_1, t_1)}{\partial(\alpha, r)} + \frac{1}{r}\frac{\partial\phi_2}{\partial\alpha}\frac{dt_0}{dr} + m_0 w_2 + m_2 w_0\right\}$$
$$-\bar{B}\left\{\left(\frac{1}{r}\frac{\partial w_1}{\partial\alpha}\right)^2 + \left(\frac{\partial w_1}{\partial r}\right)^2 + 2\frac{\partial w_2}{\partial r}\frac{dw_0}{dr}\right\}. \quad (32)$$

Integrating both sides of (32) over cross-sectional domain and then using Gauss Theorem and third equation of (22) on l.h.s., the following solution for m_2 is found:

$$m_2 = \frac{4}{RP}\cdot\frac{1541R^4}{175(288)^3}. \quad (33)$$

When expressions of m_2 and others are inserted in (32), t_2 is found in the form of

$$t_2 = t_{21}(r) + t_{22}(r)\cos 2\alpha \quad (34)$$

and two uncoupled second order ordinary differential equations are obtained from which t_{21} and t_{22} can be determined. The differential equation which corresponds to t_{21} does not involve ϕ_{22} and w_{22}. Solving it and determining the integration constants by means of (13) and (12), the following solution for t_{21} is found:

$$t_{21} = \frac{R^4}{100(288)^3}\left[\frac{F_1}{8820} + \frac{PF_2}{420} + \frac{P^2F_3}{84} + B\left(\frac{G_1}{140} + \frac{PG_2}{3} + \frac{P^2G_3}{42}\right)\right] \quad (35)$$

where

$$F_1 = 2\,926\,409 - 15\,588\,090r^2 + 28\,222\,740r^4 - 29\,194\,200r^6 + 19\,646\,550r^8 - 9\,033\,444r^{10} + 2\,769\,480r^{12} - 534\,600r^{14} + 56\,700r^{16} - 2450r^{18} \quad (36)$$

$$F_2 = 506\,917 - 2\,580\,480r^2 + 4\,339\,440r^4 - 4\,116\,840r^6 + 2\,561\,580r^8 - 1\,118\,880r^{10} + 331\,380r^{12} - 62\,640r^{14} + 6615r^{16} - 280r^{18} \quad (37)$$

$$F_3 = 301\,649 - 1\,622\,880r^2 + 3\,035\,340r^4 - 3\,365\,040r^6 + 2\,520\,630r^8 - 1\,310\,904r^{10} + 451\,080r^{12} - 96\,120r^{14} + 11\,340r^{16} - 560r^{18} \quad (38)$$

$$G_1 = 61\,033 - 598\,140r^2 + 1\,915\,200r^4 - 3\,116\,400r^6 + 2\,998\,800r^8 - 1\,811\,880r^{10} + 692\,160r^{12} - 160\,200r^{14} + 19\,800r^{16} - 980r^{18} \quad (39)$$

$$G_2 = -3082 + 7920r^2 + 11\,610r^4 - 46\,020r^6 + 55\,395r^8 - 35\,640r^{10} + 13\,860r^{12} - 3240r^{14} + 405r^{16} - 20r^{18} \quad (40)$$

and

$$G_3 = 122\,744 - 685\,440r^2 + 1\,375\,920r^4 - 1\,686\,720r^6 + 1\,422\,540r^8 - 836\,136r^{10} + 326\,340r^{12} - 78\,840r^{14} + 10\,395r^{16} - 560r^{18}. \quad (41)$$

2.4

It is of primary interest to calculate average Nusselt number, Nu. The following definition, which has appeared in many papers dealing with straight ducts, is taken:

$$Nu = \frac{2a}{K}\cdot\frac{K(\partial T/\partial\bar{r})_w}{T_{wa} - T_m}. \quad (42)$$

Under present non-dimensionalization scheme, (42) transforms to: $Nu = (2/t_{wa})$. When second order expression of t_{wa} (which is easily calculable from (21), (30) and (34)) is used, it is found that

$$Nu = 2\{t_0(1) + \lambda^2 t_{21}(1)\}^{-1}. \quad (43)$$

According to (43), t_{22} is not needed (this is the reason for our not reporting ϕ_{22} and w_{22}). Calculating $t_0(1)$ from (27) and $t_{21}(1)$ from (35) to (41) and inserting the results in (43), the Nusselt number is given by

$$Nu = \frac{6}{(1 - D^2J) + \frac{1}{2}B(1 - D^2\bar{J})} \quad (44)$$

where D is Dean number, $D = \lambda R^2$, and

$$J = \frac{1}{14(2880)^3}(34\,805 + 133\,188P + 377\,325P^2) \quad (45)$$

$$\bar{J} = \frac{1}{14(2880)^3}(3642 - 332\,640P + 595\,140P^2). \quad (46)$$

Retaining first three terms in (23) and using (25), (29) and (33), the result for longitudinal temperature gradient is presented as follows:

$$m^+ = 4(1 + B + A_1D^2),\ A_1 = \frac{1541}{175(288)^3},$$
$$m^+ = \frac{RPM}{\beta} = RPm. \quad (47)$$

It is found that

$$m^+ = 4\{1 + B + Q_0^{-1}(Q_0 - Q)\} \quad (48)$$

where Q and Q_0 denote mass flow flux across cross-section in curved and straight tubes respectively and, based on results of Section 2.2,

$$Q = Q_0(1 - A_1D^2) \quad (49)$$

Equation (48) shows, which is interesting, that in the change of straight to curved duct the increase in magnitude of longitudinal temperature gradient corresponds to the decrease in mass flow flux.

Since (44) and (47) are second order perturbation solutions, magnitude of each of JD^2, $\bar{J}D^2$ and A_1D^2 is necessarily less than unity. Keeping this in view, some qualitative observations are made in the following section.

3. DISCUSSIONS

3.1

When viscous dissipation is neglected (i.e. $B = 0$), the first equation of (47) is found as:

$$(m^+)_{B=0} = 4(1+A_1D^2). \tag{50}$$

Equation (50) shows that the magnitude of longitudinal temperature gradient (i.e. rate of rise (fall) of temperature in main flow direction in the case of wall having heat source (sink) distribution)) is larger in curved than in straight tube, and increases when Dean number increases. This may physically be interpreted as follows. The fluid movement in longitudinal direction is slower (in other words, mean velocity is lesser) in curved than in straight tube. The given wall heat flux distribution is same in both tubes. Therefore, the heat received (or given up) per unit distance in longitudinal direction by the fluid is greater in curved than in straight tube. The same is greater at higher than at lower Dean number, since mean velocity decreases as Dean number increases (which may be inferred from (49)).

From (47), the effect of viscous dissipation is to increase m^+ when B is positive (i.e. when wall contains heat source distribution) and to decrease m^+ when B is negative (i.e. when wall contains heat sink distribution). The same is found in straight duct case from

$$m_0^+ = 4(1+B). \tag{51}$$

The explanation is given as follows. Due to viscous dissipation, heat is generated in the body of fluid. Therefore, the rate of heating of fluid in longitudinal direction when β is positive would be larger (and that of cooling of fluid in that direction when β is negative would be lesser) in the presence than in the absence of viscous heating.

From (51) and (47), an interesting observation is that the positive-valued longitudinal temperature gradient effected by viscous dissipation alone remains same in curved and straight tubes. It should have been higher in curved tube due to lesser mean velocity therein. This means that heat generation due to viscous dissipation is lesser in curved than in straight tube. In fact, in straight tube, viscous heating is proportional to mean kinetic energy of fluid motion, and decreases as mean velocity decreases. Therefore, if mean velocity is reduced when tube is curved, heat generation due to viscous dissipation is likely to be reduced accordingly. This may mathematically be shown as follows. Let $\overline{W}_0$ and $\overline{W}_0-\bar{A}$, where $\bar{A} > 0$ and $\bar{A} < \overline{W}_0$, be the mean velocities in straight and curved tubes respectively. Then, in curved tube, $\partial W/\partial \bar{r}$ is of the order of $(\overline{W}_0-\bar{A})/a$ and $\partial W/\partial\alpha$ is of the order of $\bar{A}$. Therefore, from the terms involving μ in (5), the heat generation due to viscous dissipation in curved tube is of the order of

$$\mu\left\{\left(\frac{\overline{W}_0-\bar{A}}{a}\right)^2+\left(\frac{\bar{A}}{a}\right)^2\right\}. \tag{52}$$

The same in straight tube is of the order of

$$\mu\left(\frac{\overline{W}_0}{a}\right)^2. \tag{53}$$

Subtracting (53) from (52), one obtains

$$\frac{2\mu\bar{A}}{a^2}\{\bar{A}-\overline{W}_0\}. \tag{54}$$

This difference is negative, since $\bar{A} < \overline{W}_0$.

One may also imagine that in going from straight to curved duct, the viscous heating is not altered but a heat sink distribution is created in fluid medium to nullify the effect of mean velocity reduction on the longitudinal temperature gradient effected by viscous dissipation alone. In fact, the solution for m_2 consists of three terms. One term is nothing but right hand side of (33). The other two terms are equal and opposite. Retaining all those three terms, m^+ is given by

$$m^+ = 4+4B+4D^2A_1+4BD^2A_1-4BD^2A_1. \tag{55}$$

On the r.h.s. of (55), the fourth and fifth terms may be said to correspond to mean velocity reduction and above-imagined heat sink distribution respectively.

3.2

The Nusselt number solution when viscous dissipation is not taken into account is given by

$$(Nu)_{B=0} = \frac{6}{1-D^2J} \tag{56}$$

which is deduced by setting $B = 0$ in (44). From (45), J is positive, since Prandtl number P is non-negative, and increases as P increases. Therefore, from (56), Nusselt number is higher (i) at greater than at smaller Dean number; (ii) at greater than at smaller Prandtl number and (iii) in curved than in straight tube.

From (44), effect of viscous dissipation is to decrease Nusselt number when B is positive and increase when B is negative. This may be understood as follows. A temperature distribution attaining largest values at wall points and decreasing in directions of inward drawn normals, which is effected in the fluid medium due to wall heat source distribution, is augmented (i.e. becomes more marked), since a same kind temperature distribution is effected due to viscous dissipation. Therefore, magnitude of the temperature difference $T_{wa}-T_m$ is increased (which implies that Nusselt number is decreased). On the other hand, a temperature distribution attaining smallest values at wall points and increasing in directions of inward drawn normals, which is effected in the fluid medium due to wall heat sink distribution, is diminished (i.e. becomes less marked), since opposite to this is that due to viscous dissipation. Therefore, magnitude of $T_{wa}-T_m$ is reduced (which implies that Nusselt number is increased).

In cooling devices, B takes positive values. It is therefore of importance to investigate how the adverse effect of viscous dissipation on Nusselt number in the case of $B>0$ goes when tube is curved. In this context, it is required to examine the ratio of the Nusselt number in curved to that in straight tube, i.e. Nu/Nu_0. First, J and $\bar{J}$ are needed to be compared.

Since the coefficient of P^2 in (46) is greater than the coefficient of P^2 in (45), there exists a Prandtl number value, P_c, such that $\bar{J}>J$ whenever $P>P_c$. Clearly, P_c is equal to the greater of the two roots of the equation

$$\bar{J}-J=0. \tag{57}$$

It can easily be seen that one of the roots of this equation is negative and the other is positive. Therefore, P_c is equal to the positive root. On calculating this root is found to be

$$P_c = 2{\cdot}203 \text{ (approximately)}. \tag{58}$$

Since Prandtl number does not assume negative values, the following three statements hold good:

$$\bar{J}>J \quad \text{and} \quad J^+<1 \quad \text{whenever } P>P_c, \tag{59}$$

$$\bar{J}=J \quad \text{and} \quad J^+=1 \quad \text{whenever } P=P_c \tag{60}$$

and

$$\bar{J}<J \quad \text{and} \quad J^+>1 \quad \text{whenever } P<P_c \tag{61}$$

where

$$J^+ = (1-D^2\bar{J})/(1-D^2J). \tag{62}$$

The above-mentioned ratio Nu/Nu_0 is given by

$$\frac{Nu}{Nu_0} = \frac{2+B}{(1-D^2J)(2+BJ^+)}. \tag{63}$$

From (63), following informations are collected about the Nusselt number ratio Nu/Nu_0: (i) using (60), this ratio remains unaffected by viscous dissipation at $P=P_c$; (ii) from (59), it increases as B increases through positive values whenever $P>P_c$ (see Fig. 2); and (iii) from (61), it decreases as B increases through positive values whenever $P<P_c$ (see Fig. 3). Figures 2 and 3 exhibit the relationship between the Nusselt number ratio Nu/Nu_0 and the Dean number D at same set of values of the dissipation number B (where case of $B<0$ has also been included) but at different fixed values of the Prandtl number P, namely, $P=7$ in Fig. 2 and $P=0{\cdot}5$ in Fig. 3. One can see that the order of curves in Fig. 2 is reversed in Fig. 3, which is simply because $7>P_c$ and $0{\cdot}5<P_c$.

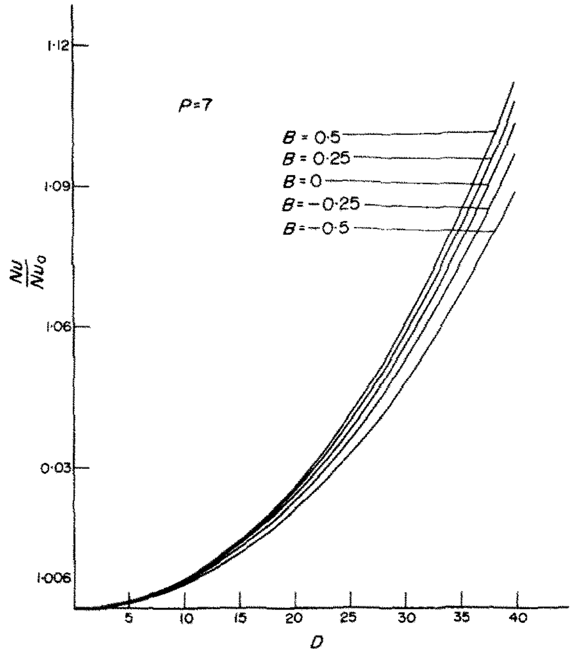

FIG. 2. $(Nu/Nu_0)_{P=7}$ vs D with B as parameter.

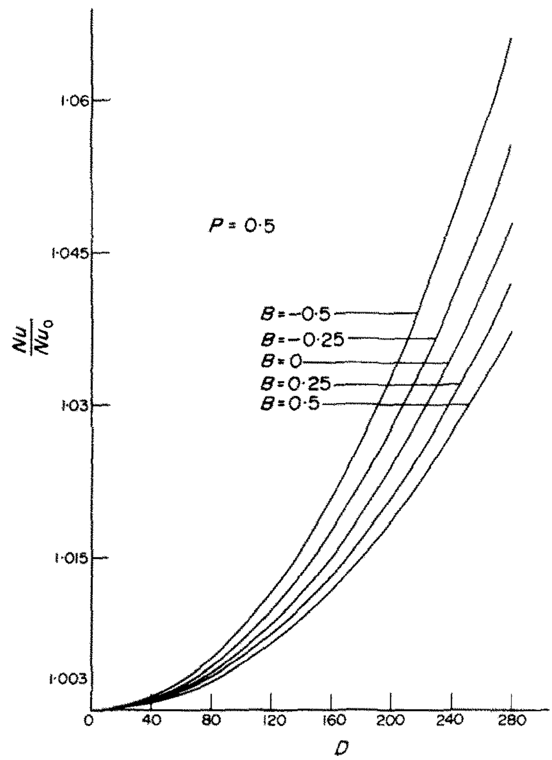

FIG. 3. $(Nu/Nu_0)_{P=0\cdot5}$ vs D with B as parameter.

3.3

The solutions which are obtained for longitudinal temperature gradient and Nusselt number without taking viscous dissipation into account have percentage errors given by

$$E_1 = 100\frac{M-(M)_{B=0}}{(M)_{B=0}} = 100\frac{m^+-(m^+)_{B=0}}{(m^+)_{B=0}},$$

$$E_2 = 100\frac{Nu-(Nu)_{B=0}}{(Nu)_{B=0}} \quad (64)$$

respectively. From (47) and (50),

$$E_1 = \frac{100B}{1+A_1D^2}. \quad (65)$$

From (44) and (56), and using (62),

$$E_2 = \frac{-100B}{2(J^+)^{-1}+B}. \quad (66)$$

From (65), it is seen that the magnitude of E_1 assumes maximum value in straight tube. This is not observed for E_2 from (66). The magnitude of E_2 goes as follows in the case of $B > 0$. At $P = P_c$, it is such as is found in straight tube, according to (60) and (66). It is lower in curved than in straight tube whenever $P > P_c$, as is concluded from (59) and (66). In view of (61) and (66), it is higher in curved than in straight tube whenever $P < P_c$.

The magnitudes of E_2 are lower for positive than for negative values of B. This can be seen in Table 1, wherein, numerical values of E_2 at few positive and negative values of B have been given.

Table 1. Few numerical values of the error E_2 at $P = P_c$

B	−0·5	−0·4	−0·3	−0·2	−0·1	0·1	0·2	0·3	0·4	0·5
E_2	33·3	25	17·6	11·1	5·3	−4·8	−9·1	−13·0	−16·7	−20·0

3.4

In the foregoing discussions, a qualitative picture of curved duct convective heat transfer has been given on the basis of the analytical results (44) and (47) and also physical explanations have been given. The quantitative picture given by (44) and (47) is not reliable for all values of the Dean number D. This is due to the fact that the second order perturbation solution for velocity field is valid only for $D \leqslant 288$, which was theoretically shown by Dean himself in [2] and experimentally verified in [12, 13]. A rigorous convergence demonstration is due to Larrain and Bonilla [16] who found that perturbation method is convergent for $D \leqslant 288$. The solution (47), therefore, holds good for $D \leqslant 288$. The Nusselt number solution (44) holds good for $D \leqslant 288$ and $DP \leqslant 288$. The validity of the present analysis is not affected by the parameter B. For instance, looking into the r.h.s. of (44) and first equation of (47), it is seen that the present solutions for Nusselt number and longitudinal temperature gradient are valid for all values of B. This is due to the fact that B occurs in straight tube case.

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APPENDIX

Under the application of Dean's simplification rules [2], mentioned in Section 2.1, the equations of motion and continuity come out as:

$$U\frac{\partial U}{\partial \bar{r}} + \frac{V}{\bar{r}}\frac{\partial U}{\partial \alpha} - \frac{V^2}{\bar{r}} - \frac{W^2\cos\alpha}{b} = -\frac{1}{\rho}\frac{\partial p}{\partial \bar{r}} - \frac{\mu}{\rho\bar{r}}\frac{\partial}{\partial \alpha}\left(\frac{\partial V}{\partial \bar{r}} + \frac{V}{\bar{r}} - \frac{1}{\bar{r}}\frac{\partial U}{\partial \alpha}\right) \quad (A.1)$$

$$U\frac{\partial V}{\partial \bar{r}} + \frac{V}{\bar{r}}\frac{\partial V}{\partial \alpha} + \frac{UV}{\bar{r}} + \frac{W^2\sin\alpha}{b} = -\frac{1}{\rho\bar{r}}\frac{\partial p}{\partial \alpha} + \frac{\mu}{\rho}\frac{\partial}{\partial \bar{r}}\left(\frac{\partial V}{\partial \bar{r}} + \frac{V}{\bar{r}} - \frac{1}{\bar{r}}\frac{\partial U}{\partial \alpha}\right) \quad (A.2)$$

$$U\frac{\partial W}{\partial \bar{r}} + \frac{V}{\bar{r}}\frac{\partial W}{\partial \alpha} = -\frac{1}{\rho b}\frac{\partial p}{\partial \theta} + \frac{\mu}{\rho}\left(\frac{\partial^2 W}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial W}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2 W}{\partial \alpha^2}\right) \quad (A.3)$$

$$\frac{\partial U}{\partial \bar{r}} + \frac{U}{\bar{r}} + \frac{1}{\bar{r}}\frac{\partial V}{\partial \alpha} = 0 \quad (A.4)$$

where p denotes pressure. These equations, after admitting (8), reduce to

$$\nabla^4\phi + \frac{1}{r}\frac{\partial(\phi, \nabla^2\phi)}{\partial(r,\alpha)} + 2\lambda w\left(\sin\alpha\frac{\partial w}{\partial r} + \frac{\cos\alpha}{r}\frac{\partial w}{\partial \alpha}\right) = 0 \quad (A.5)$$

$$\nabla^2 w + \frac{1}{r}\frac{\partial(\phi, w)}{\partial(r,\alpha)} + 4R = 0 \quad (A.6)$$

where ∇^4 denotes biharmonic operator in (r, α) and

$$R = \frac{a^3\rho}{4\mu^2}\left(-\frac{1}{b}\frac{\partial p}{\partial \theta}\right).$$

ANALYSE DU TRANSFERT THERMIQUE ETABLI ET STATIONNAIRE POUR UN ECOULEMENT LAMINAIRE DANS UNE CONDUITE CIRCULAIRE COURBE, AVEC DISSIPATION VISQUEUSE

Résumé—On considère un transfert thermique établi, stationnaire pour un écoulement laminarie à propriétés physiques constantes, dans un tube courbe, avec une distribution source de chaleur dans la paroi. On discute des conditions aux limites thermiques pour un tube courbe à section droite circulaire, en prenant en considération la génération de chaleur due à la dissipation visqueuse. Une analyse de perturbation est développée dans le cas où le gradient normal de température à la paroi est constant sur la circonférence. On obtient et discute les solutions pour le nombre de Nusselt moyen et pour le changement de température du fluide dans la direction de l'écoulement. Les effets de la dissipation visqueuse sur ces grandeurs sont étudiés. Dans la discussion, on compare les cas des tubes courbes et droits.

DIE BESTIMMUNG STATIONÄREN, VOLLSTÄNDIG AUSGEBILDETEN WÄRMEÜBERGANGES BEI LAMINARER STRÖMUNG IN GEKRÜMMTEN ROHREN MIT VISKOSER DISSIPATION

Zusammenfassung—Diese Arbeit behandelt das Problem stationären, vollständig ausgebildeten Wärmeüberganges bei laminarer Strömung mit konstanten physikalischen Werten in gekrümmten Rohren, der durch eine Wärmequellen- (oder Senken-) Verteilung in der Wand hervorgerufen wird. Die thermischen Randbedingungen werden für gekrümmte runde Rohre angegeben, wobei von der Wärmeerzeugung durch viskose Dissipation ausgegangen wird. Es wurde eine Störungsbestimmung für solche Fälle vorgenommen, bei denen der Temperaturgradient normal zur Innenwand als vorgegeben und konstant am Umfang behandelt wird. Lösungen für die mittlere *Nu*-Zahl und die Fluidtemperaturänderung in Hauptströmungsrichtung werden angegeben und erörtert. Die Auswirkung des Vorganges der viskosen Dissipation auf diese Größen wurde untersucht. Beide Fälle des gekrümmten und deraden Rohres werden miteinander verglichen.

АНАЛИЗ СТАЦИОНАРНОГО ПОЛНОСТЬЮ РАЗВИТОГО ТЕПЛООБМЕНА ПРИ ЛАМИНАРНОМ ТЕЧЕНИИ С ВЯЗКОЙ ДИССИПАЦИЕЙ В ИЗОГНУТОЙ КРУГЛОЙ ТРУБЕ

Аннотация — Рассматривается задача стационарного полностью развитого теплопереноса при ламинарном течении жидкости с постоянными физическими свойствами в изогнутой трубе при наличии источников или стоков тепла в материале стенки. Обсуждаются тепловые граничные условия для изогнутой трубы круглого сечения с учётом тепловыделения за счёт вязкой диссипации. Методом теории возмущений проанализированы случаи, когда нормальный градиент температуры внутренней стенки считается заданным и постоянным по периметру. Получены и обсуждаются решения для среднего числа Нуссельта и скорости изменения температуры жидкости в направлении основного потока. Исследуется влияние вязкой диссипации на эти величины. При обсуждении проводилось сравнение результатов для случая изогнутой и прямой трубы.